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Forward pricing in the shipping freight market

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Abstract In this paper, we derive the price of the forward freight contract using spot-forward relationship framework. We base our pricing on six different stochastic models which can capture many stylized facts of spot freight rates such as heavy-tailed logreturns, time-varying volatility and mean reversion. The models are analytically tractable which allows for pricing of forwards. We also examine the shape of forward curve for all continuous-time forward pricing formulas and find various shapes being the combination of fixed and stochastically dependent terms. Finally, this paper discusses the effect of different time to delivery and the maturity effect to the forward curve.

Keywords Freight market · Forward price · Lévy processes · Normal inverse Gaussian distribution · Stochastic volatility · Autoregressive moving average

Mathematics Subject Classification 97K60 · 60G10 · 60G51

1 Introduction

The need of efficient commodity transport between countries or continents over the world creates demand for shipping services. With the main task to transfer the commodity, a crucial part for the participants in this industry including shipowner and charterer is the cost for hiring or leasing the transportation, conceived by freight rate.

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The market for freight rate is not similar to the market of other commodities like crude oil, natural gas or agriculture. This is simply because freight services are essentially non-storable. However, the nonstorability property at least makes the shipping freight market more identical to energy markets, for instance the market for electricity and temperature. Electricity must be used once generated and we cannot trade the temperature. Therefore, the usual way for the users in freight market (electricity and temperature as well) are to enter the forward/futures contract with a specific delivery time, to ensure that commodity could be transported at that time.

Modelling the forward freight rate dynamics has been theoretically and empirically studied by many researchers. Among of them are Kavussanov and Nomikos [15,16], Koekebakker and Ådland [17] and recently in a paper by Prokopczuk [18]. To mention a few, a study done by Koekebakker and Ådland [17] used a direct Heath–Jarrow–Morton (HJM) approach to investigate the forward dynamics in a risk neutral world. They assume that the price evolution can be explained using multi-factor geometric Brownian motion. Further, four different continuous time no arbitrage pricing models have been considered in Prokopczuk [18]. The study finds that the two-factor model outperform the one-factor model based on their hedging performance.

In the present paper, we study the pricing of the forward freight rates. Our approach however is not straightforward, but based on spot-forward relationship framework. We take six different continuous stochastic models of spot freight rates introduced in Benth et al. [8] and infer forward prices thereof. To be more specific, we consider a simple geometric Brownian motion, an exponential Lévy model and a pricing model with stochastic volatility proposed by Barndorff-Nielsen and Shephard [2]. Further, we implement three different stationary continuous-time autoregressive (CAR) models in the forward pricing.

Modelling spot price of assets using geometric Brownian motion (GBM for short) is essentially assuming that their (log)returns are normally distributed. In many empirical studies (see Benth and Šaltytė Benth [4] and Benth et al. [8]), the logreturns are observed to exhibit heavy tails and also peaky in the center of distribution. Hence, the normal hypothesis is violated. Alternatively, Barndorff-Nielsen [1] proposes the normal inverse Gaussian (NIG) distribution which can be used to model the logreturns. We refer to Benth et al. [5] for details discussion on nice fitting of NIG to the empirical logreturns. Consequently, we propose to use an exponential Lévy model driving by NIG Lévy process to price forward. With the sign of volatility clustering and dependency structure exhibited in freight data (refer to Benth et al. [8]), we consider a stochastic model which incorporates stochastic volatility, namely the Barndorff-Nielsen and Shephard (hereafter BNS) stochastic volatility model. Taking into account the mean reversion property leads us to a stationary CAR(p) dynamics. We consider CAR dynamics driven by three different (Lévy) processes: Brownian motion, NIG and BNS stochastic volatility.

We price the forward with respect to a risk neutral probability. We introduce an Esscher transform (see Gerber and Shiu [12] who first applied Esscher transform to financial markets) for general Lévy processes driving the spot dynamics. The use of such transformation in energy markets can be found in Benth et al. [5]. The Esscher alters the dynamics with a constant called *market price of risk* such that the price process becomes martingale after discounting. We also introduce the Esscher transform to the stochastic volatility dynamics that would exponentially tilt the Lévy jump measure by a coefficient called *market price of volatility risk*.

In pricing forward contracts for seasonally dependent commodities, the shape of the forward curve is depending on the demand and supply which varies seasonally (see Borovkova and Geman [9] for the discussion of seasonality behaviour in the forward curve). Electricity for example are highly demanded during winter season for heating compared to summer. The temperature market is obviously seasonally dependent. However, the failure of supply side to react quickly to meet the demand, distinguishing freight market from other seasonal-dependent market. A simple fitting using seasonal mean function in Benth et al. [8] shows no (deterministic) seasonality in the dry bulk freight time series. In addition, a seasonality study of tanker market segment by Kavussanov and Alizadeh [14] has rejected the existence of stochastic seasonality in freight rates. Since the findings of seasonality are mixed, we decided to neglect seasonality in our forward pricing.

We present our findings as follows. In Sect. 2, we introduce the stochastic dynamics of spot price that shall be used in forward pricing. Next, the forward prices are derived in Sect. 3 where the Esscher transform for risk-neutral pricing measure is implemented. Section 4 discusses various shapes of forward curves for different stochastic models. Finally, Sect. 5 concludes the paper.

2 Stochastic dynamics of the spot price

This section describes the six different stochastic spot models which were introduced in Benth et al. [8]. We shall use all the models for our forward pricing. Now, let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which satisfies the usual conditions (see Barndorff-Nielsen and Shiryaev [3]).

2.1 Geometric Brownian motion

Denote $S(t)$ as the spot price at time $t \geq 0$. A geometric Brownian motion model explains the spot using the dynamics

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \tag{2.1}$$

where the constant μ and $\sigma > 0$ are respectively the drift and volatility and $B(t)$ is Brownian motion. Solving (2.1) for $T \geq t \geq 0$ yields

$$S(T) = S(t)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\int_t^T dB(u)\right). \tag{2.2}$$

The model assumes that the logreturns will be independent and normally distributed. The empirical distribution of the logreturns for freight rates studied in Benth et al. [8] shows the concentration of mass in the center of the distribution. Moreover, the tails are far more heavy than normal. Thus, the hypothesis of normally distributed logreturns should be rejected and ultimately, GBM is not an appropriate model for the spot.

Instead of using GBM to model the spot price, one could use an exponential Lévy model which generalize the Brownian motion to Lévy process. The model allows for jumps and the price path would be no more continuous. A possible candidate which can explain the spot is the normal inverse Gaussian Lévy process. The study by Benth et al. [8] shows that NIG fitted the logreturns of the freight rates very well, capturing the tails and the high peak in the center. Next, we introduce the exponential Lévy model that will be used in our forward pricing.

2.2 Lévy-based dynamics

The spot price $S(t)$ is defined as an exponential Lévy process which takes the form

$$S(t) = S(0)\exp(L(t)). \tag{2.3}$$

We choose $L(t)$ to be NIG Lévy process, that is a stochastic process with normal inverse Gaussian distributed increments. Therefore, we called (2.3) as NIG Lévy model. We explain briefly the NIG distribution herein and details can be found in Benth et al. [5]. The NIG with four parameters α, β, δ and μ is a class of generalized hyperbolic distributions having the density functions,

$$f(x;\alpha,\beta,\delta,\mu) = k\exp(\beta(x-\mu))\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}},$$

where $k = \delta\alpha\exp(\delta\sqrt{\alpha^2-\beta^2})/\pi$ and $K_1(x)$ is the modified Bessel function of the third kind with index 1.

We may classify $L(t)$ as NIG Lévy process if $L(1)$ is distributed according to the normal inverse Gaussian distribution. The Lévy measure of $L(t)$ is given by (see Barndorff-Nielsen and Shephard [2])

$$\ell(dz) = \frac{\alpha\delta}{\pi|z|}e^{\beta z}K_1(\alpha|z|)dz, \tag{2.4}$$

and the cumulant function is given by (see Benth and Šaltytė-Benth [4])

$$\psi(\lambda) = i\lambda\mu + \delta\left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+i\lambda)^2}\right). \tag{2.5}$$

In Benth et al. [8], the time series of empirical logreturns of freight rates show volatility clustering. This fact together with heavy-tailed logreturns give signs of stochastic volatility in the dynamics. Following the study, we propose to use the BNS model to describe the time-varying volatility process.

2.3 Barndoff-Nielsen and Shephard stochastic volatility model

In energy markets, for example electricity, it is rather natural to assume that the volatility of the price process is changing stochastically over time. The empirical studies in

Benth [5,6] show that the use of stochastic volatility model may capture many stylized facts of the empirical logreturns data, and he proposed to use the BNS model. We shall follow his step by defining the BNS stochastic volatility model with the following.

Denote $X(t) = \ln S(t)$ as the solution of the stochastic differential equation,

$$dX(t) = \{\mu + \beta\sigma^2(t)\}\,dt + \sigma(t)\,dB(t), \tag{2.6}$$

where $B(t)$ is standard Brownian motion. The stationary volatility process, $\sigma^2(t)$ follows a weighted sum of processes $V_j(t)$ given by

$$\sigma^2(t) = \sum_{j=1}^{n} \omega_j V_j(t). \tag{2.7}$$

The weights $\omega_j \in [0, 1]$ for $j = 1, \ldots, n$ are summing up to one. Meanwhile, the dynamics of the Ornstein–Uhlenbeck process $V_j(t)$ takes the form

$$dV_j(t) = -\lambda_j V_j(t)\,dt + dZ_j(\lambda_j t), \tag{2.8}$$

where $\lambda > 0$ is the speed of mean reversion of the volatility process. The process $Z(\lambda t)$ is called a subordinator, that is the process with only positive increments and no drift. This ensures the positivity of variable $V(t)$. We let $V(t)$ follow the inverse Gaussian law, therefore the increments of (2.6) would approximately be NIG distributed.

For $T \geq t \geq 0$, we can reformulate Eq. (2.6) to be

$$S(T) = S(t)\exp\left(\mu(T-t) + \beta\int_t^T \sigma^2(u)\,du + \int_t^T \sigma(u)dB(u)\right). \tag{2.9}$$

2.4 CAR(p) dynamics

Following Schwartz [19], we assume that the spot freight rates are characterized by an exponential Ornstein–Uhlenbeck stochastic process. We choose to work with a stationary model of continuous time autoregressive (CAR), a subclass of *continuous autoregressive moving average* (CARMA) model (see Brockwell [10]). The motivation is mainly triggered by empirical evidence of freight rates in Benth et al. [8] that the prices are mean-reverting and showing dependency structure.

By working with one-dimensional Brownian motion $B(t)$, we define a CAR(p) dynamics as follows. For $p \geq 1$, the p-dimensional Ornstein–Uhlenbeck stochastic process $\mathbf{X}(t)$ is defined as the solution of

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \mathbf{e}_p\sigma\,dB(t), \tag{2.10}$$

with A is the $p \times p$ matrix given by

$$A = \begin{bmatrix} \mathbf{0} & I \\ -\alpha_p \;\ldots & -\alpha_1 \end{bmatrix}, \tag{2.11}$$

where $\mathbf{0}$ is the $(p-1)$ vector of zeros and I represents the $(p-1)\times(p-1)$ identity matrix. The matrix A contains positive constants α_i for $i=1,\ldots,p$, corresponding to the different speed of mean reversions. Moreover, $\{\mathbf{e}_i\}_{i=1}^{p}$ is the i-th canonical basis vectors in $\mathbb{R}^p$ and σ is a (constant) volatility. By Itô formula, the explicit solution of (2.10) for $s \geq t$ is given as

$$\mathbf{X}(s) = e^{A(s-t)}\mathbf{X}(t) + \int_t^s e^{A(s-u)}\mathbf{e}_p \sigma\, dB(u). \tag{2.12}$$

Setting the log price $\ln S(t) = Y(t)$, a CAR(p) is now defined as $Y(t) = \mathbf{e}_1'\mathbf{X}(t)$.

Looking back to the analysis of spot freight rates in Benth et al. [8], the observed residuals of the log price after removing the autoregressive effects essentially come from nonGaussian distribution. They fitted the empirical density with NIG class distribution, and the distribution looks very well fitted. Therefore, the study proposes an alternative CAR model having similar structure as in (2.10) but with L being Lévy process with NIG distributed increments. The dynamics of $\mathbf{X}(t)$ takes the form

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \mathbf{e}_p\,dL(t). \tag{2.13}$$

In this sense, the volatility takes value one, that is $\sigma(t) = 1$.

The analysis of residuals also shows the sign of stochastic volatility, where the tails of residuals distribution are heavier than normal and there are dependency structures indicated in the log price. The CAR with BNS stochastic volatility is defined as follows

$$d\mathbf{X}(t) = A\mathbf{X}(t)\,dt + \mathbf{e}_p\,\sigma(t)\,dB(t). \tag{2.14}$$

By having normally distributed stochastic process $B(t)$ and $\sigma(t) \sim IG(\delta,\gamma)$, $Y(t)$ would be approximately NIG distributed.

3 Pricing of freight forwards

In this section, we shall derive the forward pricing formulas for various spot models, as described in the previous section. Denote $F(t,T)$ as the price of a forward contract at time t with delivery time T where $0 \leq t \leq T < \infty$. We define $F(t,T)$ to be the expected spot price at time T conditional on information revealed up to time t, being priced under risk neutral probability $Q \sim P$. This can be mathematically expressed as,

$$F(t,T) = \mathbb{E}_Q[S(T)|\mathcal{F}_t]. \tag{3.1}$$

It is assumed that $F(t,T)$ is $\mathcal{F}_t$-adapted and $S(T) \in L^1(Q)$ which makes the arbitrage-free property holds.

Pricing forwards requires a pricing probability Q. Following Benth et al. [5], we introduce a parametric class of measure change of Girsanov transform for the case of Gaussian model using

$$B_\theta(t) = B(t) - \theta t, \tag{3.2}$$

with θ as a constant describing the *market price of risk*. Using this transformation, the Q-dynamics of (2.1) is now taking the form

$$dS(t) = \kappa S(t)dt + \sigma S(t)dB_\theta(t), \tag{3.3}$$

where $\kappa = \mu + \sigma\theta$ and B_θ is a Q-Brownian motion. The explicit solution of (3.3) is

$$S(T) = S(t)\exp\left(\left(\kappa - \frac{\sigma^2}{2}\right)(T-t) + \int_t^T \sigma dB_\theta(u)\right). \tag{3.4}$$

Now, we are ready to derive the forward pricing formula under geometric Brownian motion model.

Proposition 3.1 *The price at time t for a forward contract with delivery at time* $T \geq t \geq 0$ *under geometric Brownian motion model is given as*

$$F(t,T) = S(t)\exp\left(\kappa\,(T-t)\right), \tag{3.5}$$

where $\kappa = \mu + \sigma\theta$.

Proof From definition and with appealing to (3.4) we have

$$\begin{aligned} F(t,T) &= \mathbb{E}_Q[S(T)|\mathcal{F}_t] \\ &= S(t)\exp\left(\left(\kappa - \frac{\sigma^2}{2}\right)(T-t)\right)\cdot \mathbb{E}_Q\left[\exp\left(\int_t^T \sigma dB_\theta(s)\right)\middle|\mathcal{F}_t\right]. \end{aligned}$$

We know that the stochastic integral is independent of $\mathcal{F}_t$ and $B_\theta \sim \mathcal{N}(0, T-t)$. Hence, the proposition follows from independent increment property of Brownian motion. ◻

Moving on from a specific class of Brownian motion to the more general Lévy processes in deriving risk-neutral forward price requires a more flexible change of measure. Apparently, this can be done using the Esscher transform which can be considered as generalization of Girsanov transformation. Historically, it has been introduced by Esscher [11] in approximating the aggregated claim distribution, so that the chosen parameter θ drifting the mean to a new point of interest. This density transformation has later been developed by Gerber and Shiu [12] to pricing options and recently it has been widely used in forward and futures pricing (see e.g. Benth et al. [5], and Benth and Sgarra [7]).

The next proposition will be devoted to the derivation of explicit forward price for NIG Lévy model. In order to do so, we state here some possible results from Benth and Šaltytė Benth [4] concerning the existence of at least the first moment of the spot process by imposing an exponential integrability condition on the Lévy measure as follows.

Condition 1 *There exists a constant $k > 0$ such that the Lévy measure satisfies the integrability condition*

$$\int_1^\infty e^{kz}\ell(dz) < \infty.$$

The order of the moments is finite, and determined by the constant k. The following Lemma ensure the finite moment condition exists.

Lemma 3.2 *If $g : [0, t] \mapsto \mathbb{R}$ is a bounded and measurable function and Condition 1 holds for $k := \sup_{s\in[0,t]} |g(s)|$, then*

$$\mathbb{E}\left[\exp\left(\int_0^t g(u)\, dL(u)\right)\right] = \exp\left(\int_0^t \phi(g(u))du\right),$$

where $\phi(\lambda) = \psi(-i\lambda)$.

Proof The proof can be found in Benth and Šaltytė Benth [4]. □

Let us consider a constant θ_L to be the *market price of risk*. For $0 \leq t \leq T$, we define a process $\pi_L(t)$ as

$$\pi_L(t) = \exp\left(\theta_L L(t) - \phi_L(\theta_L)t\right).$$

Here, ϕ_L is logarithm of the moment generating function of Lévy process L (or sometimes called the cumulant function). Thus, we define the Radon-Nikodym derivative

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \pi_L(t),$$

such that π_L is the density process of a measure $Q \sim P$. The next proposition will be the price formula for forward using NIG Lévy spot model.

Proposition 3.3 *The price at time t for a forward contract with delivery at time $T \geq t \geq 0$ under NIG Lévy model is given as*

$$F(t, T) = S(t)\exp\left(\{\phi_L(\theta_L + 1) - \phi_L(\theta_L)\}(T - t)\right).$$

Proof Using definition and from (2.3) we have,

$$\begin{aligned}\mathbb{E}_Q\left[S(T) \mid \mathcal{F}_t\right] &= S(0)e^{L(t)}\mathbb{E}_Q\left[e^{L(T)-L(t)} \mid \mathcal{F}_t\right]\\ &= S(t)\cdot\mathbb{E}_Q\left[\exp\left(\int_t^T 1\, dL(u)\right)\middle|\mathcal{F}_t\right].\end{aligned}$$

Appealing Bayes' Formula to the conditional expectation, we have

$$\begin{aligned}
&\mathbb{E}_Q\left[\exp\left(\int_t^T 1\,dL(u)\right)\Bigg|\mathcal{F}_t\right]\\
&=\mathbb{E}\left[\exp\left(\int_t^T 1\,dL(u)\right)\frac{\pi_L(T)}{\pi_L(t)}\Bigg|\mathcal{F}_t\right]\\
&=\exp\left(-\int_t^T \phi_L(\theta_L)du\right)\mathbb{E}\left[\exp\left(\int_t^T (\theta_L+1)\,dL(u)\right)\Bigg|\mathcal{F}_t\right]\\
&=\exp\left(-\int_t^T \phi_L(\theta_L)du\right)\mathbb{E}\left[\exp\left(\int_t^T (\theta_L+1)\,dL(u)\right)\right]\\
&=\exp\left(\int_t^T \{\phi_L(\theta_L+1)-\phi_L(\theta_L)\}\,du\right),
\end{aligned}$$

from independent increment of Lévy process and introducing the cumulant function. This proves the proposition. □

Taking into account the stochastic volatility in the price dynamics makes the forward pricing dependent on the path of volatility process and to compute the price now is not a direct attainable task. Under risk neutral pricing measure Q, one could obtain the forward price by transforming the density of the Lévy process using Esscher transform as what we did for pricing forward under NIG Lévy model. Furthermore, we can also transform the Lévy density of the volatility process by introducing *market price of volatility risk* into the dynamics. We will focus here the BNS stochastic volatility model.

Let θ_V be the market price of volatility risk, we define a process π_Z with

$$\pi_Z(t)=\exp\left(\theta_V Z(t)-\phi_Z(\theta_V)t\right).$$

To this end, we introduce density process $\pi(t)$ to be the product of $\pi_L(t)$ and $\pi_Z(t)$, mathematically expressed as

$$\frac{dQ}{dP}\bigg|_{\mathcal{F}_t}=\pi_L(t)\times\pi_Z(t)=\pi(t).$$

We formulate the forward price for BNS stochastic volatility model in the next proposition.

Proposition 3.4 *The price at time t for a forward contract with delivery at time $T\geq t\geq 0$ under BNS stochastic volatility model is given as*

$$\begin{aligned}
F(t,T)=S(t)\exp&\left((\mu+\theta)(T-t)+\sum_{j=1}^n\Theta(T-t)V_j(t)\right)\\
&\times\exp\left(\sum_{j=1}^n\int_t^T\{\phi_Z\left(\Theta(T-\nu)+\theta_V\right)-\phi_Z(\theta_V)\}\,d\nu\right),
\end{aligned}$$

where $\Theta(\xi)=\frac{\omega_j}{\lambda_j}(\beta+0.5)\left(1-e^{-\lambda_j\xi}\right)$.

Proof We obtain the risk-neutral dynamics of (2.6),

$$dX(t)=\{\mu+\theta+\beta\sigma^2(t)\}\,dt+\sigma(t)\,dB_\theta(t), \tag{3.6}$$

by introducing parametric class of risk-neutral probabilities using Girsanov transform,

$$dB_\theta(t)=dB(t)-\frac{\theta}{\sigma(t)}\,dt. \tag{3.7}$$

We know that $\ln S(t)=X(t)$. By definition and reformulating (3.6), we have

$$\begin{aligned}\mathbb{E}_Q\left[S(T)\mid\mathcal{F}_t\right]&=S(t)\exp\left((\mu+\theta)(T-t)\right)\\&\quad\times\mathbb{E}_Q\left[\exp\left(\int_t^T\beta\sigma^2(u)du+\int_t^T\sigma(u)dB_\theta(u)\right)\middle|\mathcal{F}_t\right].\end{aligned} \tag{3.8}$$

Observe that the transformation of probability from P to Q does not alter the characteristics of $\sigma(t)$. Using the same argument for Proposition 2.2 in Benth [6], we introduce the σ-algebra

$$\mathcal{F}_t^\sigma=\sigma\{\sigma^2(u),0\le u\le t\}\bigvee\mathcal{F}_t.$$

Using the tower property of conditional expectations, and from the independence between $\sigma(t)$ and $B_\theta(t)$ and the independent increment property of Brownian motion yields

$$\begin{aligned}&\mathbb{E}_Q\left[\exp\left(\int_t^T\beta\sigma^2(u)du+\int_t^T\sigma(u)dB_\theta(u)\right)\middle|\mathcal{F}_t\right]\\&=\mathbb{E}_Q\left[\exp\left(\int_t^T\beta\sigma^2(u)du\right)\mathbb{E}\left[\exp\left(\int_t^T\sigma(u)dB_\theta(u)\right)\mid\mathcal{F}_t^\sigma\right]\middle|\mathcal{F}_t\right]\\&=\mathbb{E}_Q\left[\exp\left(\left(\beta+\frac{1}{2}\right)\int_t^T\sigma^2(u)du\right)\middle|\mathcal{F}_t\right].\end{aligned}$$

Now, we compute the conditional expectation. We have

$$\begin{aligned}&\mathbb{E}_Q\left[\exp\left(\left(\beta+\frac{1}{2}\right)\int_t^T\sigma^2(u)du\right)\middle|\mathcal{F}_t\right]\\&=\mathbb{E}_Q\left[\exp\left(\left(\beta+\frac{1}{2}\right)\int_t^T\sum_{j=1}^n\omega_jV_j(u)\,du\right)\middle|\mathcal{F}_t\right]\\&=\prod_{j=1}^n\mathbb{E}_Q\left[\exp\left(\omega_j\left(\beta+\frac{1}{2}\right)\int_t^TV_j(u)\,du\right)\middle|\mathcal{F}_t\right],\end{aligned}$$

after replacing $\sigma(u) = \sum_{j=1}^{n} \omega_j V_j(u)$ and the last equality follows from independence of the stochastic volatility factors $V_j(u)$. We shall replace $V_j(u)$ in the conditional expectation with the following explicit dynamics of stochastic volatility

$$V_j(u) = e^{-\lambda_j(u-t)} V_j(t) + \int_t^u e^{-\lambda_j(u-\nu)} dZ_j(\nu). \tag{3.9}$$

By adaptedness of $V(t)$, we have

$$\begin{aligned}
&\mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T V_j(u)\, du \right) \mid \mathcal{F}_t \right] \\
&= \exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T e^{-\lambda_j(u-t)} du\, V_j(t) \right) \\
&\quad \times \mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T \int_t^u e^{-\lambda_j(u-\nu)}\, dZ_j(\nu)\, du \right) \Big| \mathcal{F}_t \right].
\end{aligned}$$

Appealing the stochastic Fubini theorem to the conditional expectation and with exponential integrability conditions, we have

$$\begin{aligned}
&\mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T \int_t^u e^{-\lambda_j(u-\nu)}\, dZ_j(\nu)\, du \right) \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T \int_t^T \mathbf{1}_{(\nu \le u)} e^{-\lambda_j(u-\nu)} dZ_j(\nu) du \right) \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T \int_t^T \mathbf{1}_{(\nu \le u)} e^{-\lambda_j(u-\nu)} du\, dZ_j(\nu) \right) \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T \int_\nu^T e^{-\lambda_j(u-\nu)} du\, dZ_j(\nu) \right) \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}_Q \left[\exp\left(\omega_j \left(\beta + \frac{1}{2} \right) \int_t^T \frac{1}{\lambda_j} \left(1 - e^{-\lambda_j(T-\nu)} \right) dZ_j(\nu) \right) \Big| \mathcal{F}_t \right] \\
&= \exp\left(\int_t^T \left\{ \phi_Z \left(\frac{\omega_j}{\lambda_j} \left(\beta + \frac{1}{2} \right) \left(1 - e^{-\lambda_j(T-\nu)} \right) + \theta_V \right) - \phi_Z(\theta_V) \right\} d\nu \right),
\end{aligned}$$

follows from independent increment of Lévy process for stochastic volatility. Combining all terms yields the result. □

Many empirical studies validated the presence of memory structure in the spot proces of energy related market including freight (see Benth et al. [8]). In the previous section, we define stationary CAR(p) dynamics from there we can derive the forward pricing formula. In the next proposition, we compute the forward price based on CAR dynamics where the residuals are normally distributed with zero mean.

Proposition 3.5 *The price of a forward contract at time t for delivery at time* $T \geq t \geq 0$ *under CAR(p) model driven by Brownian motion is given as*

$$F(t,T) = \exp\left(\mathbf{e}_1' e^{A(T-t)}\mathbf{X}(t) + \int_t^T \Sigma(T-u)\theta\sigma\, du\right) \times \exp\left(\frac{1}{2}\int_t^T \Sigma^2(T-u)\sigma^2 du\right),$$

where $\Sigma(T-u) = \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p$.

Proof Given $S(t) = e^{Y(t)}$. We obtain the explicit dynamics of risk-neutral CAR(p) (2.10) after introducing Girsanov transform (3.2) as

$$Y(T) = \mathbf{e}_1' e^{A(T-t)}\mathbf{X}(t) + \int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\, \sigma\theta\, du + \int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\, \sigma\, dB_\theta(u). \tag{3.10}$$

By definition and inserting (3.10), we get

$$\begin{aligned} F(t,T) &= \mathbb{E}_Q\left[S(T) \mid \mathcal{F}_t\right] \\ &= \exp\left(\mathbf{e}_1' \mathrm{e}^{A(T-t)}\mathbf{X}(t) + \int_t^T \mathbf{e}_1' \mathrm{e}^{A(T-u)}\mathbf{e}_p\, \theta\sigma\, du\right) \\ &\quad \times \mathbb{E}_Q\left[\exp\left(\int_t^T \mathbf{e}_1' \mathrm{e}^{A(T-u)}\mathbf{e}_p\, \sigma dB_\theta(u)\right)\middle|\mathcal{F}_t\right]. \end{aligned} \tag{3.11}$$

Using the same argument as in the proof of Proposition 3.4, we obtain

$$\begin{aligned} &\mathbb{E}_Q\left[\exp\left(\int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\, \sigma dB_\theta(u)\right)\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(\frac{1}{2}\int_t^T (\mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p)^2\sigma^2 du\right)\middle|\mathcal{F}_t\right] \\ &= \exp\left(\frac{1}{2}\int_t^T (\mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p)^2\sigma^2 du\right), \end{aligned}$$

from independent increment property of B_θ. The proposition follows after change of variables. □

As indicated, the continuous stochastic process of Brownian motion is not appropriate to explain the evolution of CAR process (see Benth et al. [8]). Thus, we propose in the previous section the NIG process driving the CAR dynamics which is believed to be the potential Lévy process to capture the distributional properties of residuals. We will here use the CAR dynamics driven by NIG process as in (2.13) to formulate the forward price in the next proposition.

Proposition 3.6 *The price of a forward contract at time t for delivery at time $T \geq t \geq 0$ under CAR(p) model driven by normal inverse Gaussian process is given as*

$$F(t,T) = \exp\left(\mathbf{e}_1' e^{A(T-t)}\mathbf{X}(t) + \int_t^T \{\phi_L(\Sigma(T-u)+\theta_L) - \phi_L(\theta_L)\}\, du\right),$$

where $\Sigma(T-u) = \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p$.

Proof We have the explicit solution of (2.13) as

$$Y(T) = \mathbf{e}_1' e^{A(T-t)}\mathbf{X}(t) + \int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\, dL(u). \tag{3.12}$$

By definition and inserting (3.12), we obtain

$$\begin{aligned}\mathbb{E}_Q[S(T)|\mathcal{F}_t] &= \exp(\mathbf{e}_1' e^{A(T-t)}\mathbf{X}(t)) \\ &\quad \times \mathbb{E}_Q\left[\exp\left(\int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\, dL(u)\right)\middle|\mathcal{F}_t\right].\end{aligned}$$

Using a short notation $\Sigma(T-u) = \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p$ and by appealing Bayes' formula to the conditional expectation part, we get

$$\begin{aligned}&\mathbb{E}_Q\left[\exp\left(\int_t^T \Sigma(T-u)dL(u)\right)\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\exp\left(\int_t^T \Sigma(T-u)dL(u)\right)\frac{\pi_L(T)}{\pi_L(t)}\middle|\mathcal{F}_t\right] \\ &= \exp\left(-\int_t^T \phi_L(\theta_L)du\right)\mathbb{E}\left[\exp\left(\int_t^T (\Sigma(T-u)+\theta_L)dL(u)\right)\middle|\mathcal{F}_t\right] \\ &= \exp\left(\int_t^T \{\phi_L(\Sigma(T-u)+\theta_L) - \phi_L(\theta_L)\}du\right).\end{aligned}$$

Hence, the proposition follows. □

Our next proposition will be the forward pricing under CAR dynamics driven by BNS stochastic volatility process. The forward price is now dependent to the stochastic volatility $\sigma^2(t)$.

Proposition 3.7 *The price of a forward contract at time t for delivery at time $T \geq t \geq 0$ under CAR(p) driven by BNS stochastic volatility process is given as*

$$F(t,T) = \exp\left(\mathbf{e}_1' \mathrm{e}^{A(T-t)}\mathbf{X}(t) + \int_t^T \Sigma(T-u)\theta du\right)$$
$$\times \exp\left(\sum_{j=1}^n \frac{\omega_j}{2} \int_t^T \Sigma^2(T-u)e^{-\lambda_j(u-t)}\, du V_j(t)\right)$$
$$\times \exp\left(\sum_{j=1}^n \int_t^T \left\{\phi_Z\left(\frac{\omega_j}{2}\int_v^T \Upsilon(u-v)\, du + \theta_V\right) - \phi_Z(\theta_V)\right\} dv\right),$$

where $\Upsilon(x) = \Sigma^2(T-u)e^{-\lambda_j x}$*, and* $\Sigma(T-u) = \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p$*.*

Proof We obtain

$$Y(T) = \mathbf{e}_1' e^{A(T-t)}\mathbf{X}(t) + \int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\theta\, du + \int_t^T \mathbf{e}_1' e^{A(T-u)}\mathbf{e}_p\sigma(t)\, dB_\theta(u), \tag{3.13}$$

after appealing the Girsanov transform (3.7) and solving (2.14). By definition,

$$F(t,T) = \mathbb{E}_Q\left[S(T)\middle|\mathcal{F}_t\right]$$
$$= \exp\left(\mathbf{e}_1' \mathrm{e}^{A(T-t)}\mathbf{X}(t) + \int_t^T \mathbf{e}_1' \mathrm{e}^{A(T-u)}\mathbf{e}_p\theta du\right)$$
$$\times \mathbb{E}_Q\left[\exp\left(\int_t^T \mathbf{e}_1' \mathrm{e}^{A(T-u)}\mathbf{e}_p\sigma(u)dB_\theta(u)\right)\middle|\mathcal{F}_t\right].$$

Consider the conditional expectation part. Using the same argument as in the proof of Proposition 3.4, we get

$$\mathbb{E}_Q\left[\mathbb{E}_Q\left[\exp\left(\int_t^T \mathbf{e}_1' \mathrm{e}^{A(T-u)}\mathbf{e}_p\sigma(u)dB_\theta(u)\right)\middle|\mathcal{F}_t^\sigma\right]\middle|\mathcal{F}_t\right]$$
$$= \mathbb{E}_Q\left[\exp\left(\frac{1}{2}\int_t^T (\mathbf{e}_1' \mathrm{e}^{A(T-u)}\mathbf{e}_p)^2\sigma^2(u)du\right)\middle|\mathcal{F}_t\right]$$
$$= \prod_{j=1}^n \mathbb{E}_Q\left[\exp\left(\frac{\omega_j}{2}\int_t^T \Sigma^2(T-u)V_j(u)du\right)\middle|\mathcal{F}_t\right]$$
$$= \prod_{j=1}^n \exp\left(\frac{\omega_j}{2}\int_t^T \Sigma^2(T-u)e^{-\lambda_j(u-t)}du\, V_j(t)\right)$$
$$\times \mathbb{E}_Q\left[\exp\left(\frac{\omega_j}{2}\int_t^T\int_t^u \Sigma^2(T-u)e^{-\lambda_j(u-v)}dZ_j(v)du\right)\middle|\mathcal{F}_t\right],$$

after replacing the dynamics of stochastic volatility (3.9) and considering the independency of stochastic volatility factor $V_j(u)$. Invoking the stochastic Fubini theorem to the conditional expectation of the last equality and from Bayes' formula, we have

$$\begin{aligned}
&\mathbb{E}_Q\left[\exp\left(\frac{\omega_j}{2}\int_t^T\int_t^u \Sigma^2(T-u)e^{-\lambda_j(u-v)}dZ_j(v)du\right)\Big|\mathcal{F}_t\right]\\
&\quad=\mathbb{E}_Q\left[\exp\left(\frac{\omega_j}{2}\int_t^T\int_v^T \Sigma^2(T-u)e^{-\lambda_j(u-v)}dudZ_j(v)\right)\Big|\mathcal{F}_t\right]\\
&\quad=\exp\left(\int_t^T\left\{\phi_Z\left(\theta_V+\frac{\omega_j}{2}\int_v^T \Sigma^2(T-u)e^{-\lambda_j(u-v)}du\right)-\phi_Z(\theta_V)\right\}dv\right).
\end{aligned}$$

Combining all terms yields the result. □

We have now completed the derivation of forward price for freight market using all six models of spot freight rates discovered by Benth et al. [8]. In the following section, we will discuss the various shape of forward curve contributed by many factors in the explicit forward formulas.

4 Shapes of the forward curves

The price of a forward contract brings information on the behaviour of the spot price in the future. We shall examine here the shape of the forward curve based on explicit forward formulas derived from different spot models. These consist of GBM, NIG Lévy, BNS stochastic volatility and CAR model with Brownian motion, NIG and BNS stochastic volatility distributed increments. Let focus on the first three explicit forward prices in Proposition 3.1, 3.3 and 3.4 respectively under GBM, NIG Lévy and BNS stochastic volatility spot models. In general, the shape of the forward curve is determined by today's spot price, $S(t)$ and a contribution from some constant including risk premium, θ in the exponential which give rise to a fixed shape. We can simply see that for forward under GBM model, the curve will be exponentially increasing or decreasing respectively called contango or backwardation depending on the positive or negative value of κ. For the forward price under NIG Lévy spot model, additional contribution comes from integrands of cumulant function. We can directly compute the integral with having the cumulant function of normal inverse Gaussian as in (2.5). The forward for BNS stochastic volatility spot model however contains the contribution from stochastic volatility (in the second term of Proposition 3.4) which is weighted by $\omega_j(\beta+0.5)\{e^{-\lambda_j(u-t)}-1\}/\lambda_j$. The last term involving the integrands of cumulant function of inverse Gaussian which is not stochastically varying.

The most important part for the forward price under BNS stochastic volatility spot model is time-varying volatility $V(t)$, appearing in Proposition 3.4 which also accounts for the random changes in the price. Thus, one need to have the current state of stochastic volatility to figure out the shape of the forward. This can be simulated using a series representation (see Barndorff-Nielsen and Shephard [2]) or an exact scheme proposed by Zhang and Zhang [20]. In Fig. 1, we have plotted the forward

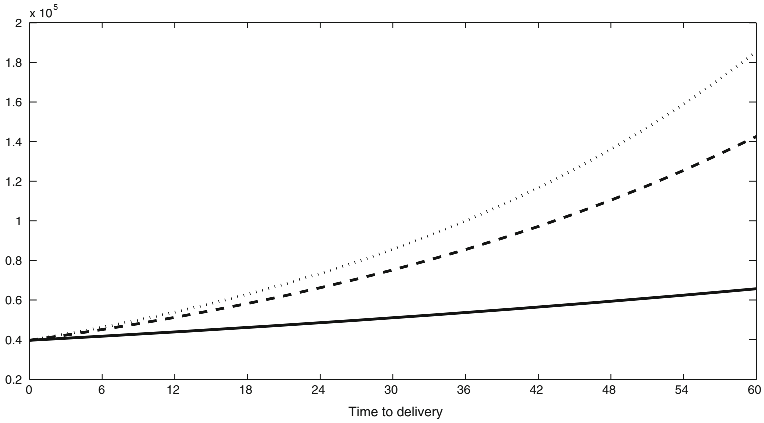

Fig. 1 Forward prices at $t = 0$ under GBM (*complete line*), NIG Lévy (*dashed line*) and BNS stochastic volatility (*dotted line*) spot models with $S(0) = 39663$

curve at $t = 0$ under GBM spot model together with the curves for NIG Lévy and BNS stochastic volatility spot models for different time to delivery using parameters in Benth et al. [8]. However, we are not going to input any risk premium here, thus setting $\theta = \theta_L = \theta_V = 0$. The effect of risk premium is just scaling up or down the original curve.

Obviously, the curve for forward prices under GBM is slowly increasing with increasing time to delivery. The curves for the other two models deviates quite fast with large deviation for longer time to delivery. In addition, the forward curve under BNS stochastic volatility model shows significantly higher level since longer time to delivery increase the volatility effect which affecting prices in the long end.

Now, we investigate the effect of time to delivery for these three models. Assume that we have a contract with fixed delivery time, say $T = 252$. The forward price $t \to F(t, T)$ under GBM, NIG Lévy and BNS stochastic volatility spot models are plotted in Fig. 2. Starting from forward prices at $t = 0$, we let time to move forward until delivery and see the price difference between the models. From the discussion above and looking at Fig. 1, there are huge differences in the forward price under GBM with NIG Lévy and BNS stochastic volatility at $t = 0$ and in Fig. 2, we can still see the difference within the first 50 days.

The path for NIG Lévy and BNS stochastic volatility looks similar and not much different after 50 days, but they differ much with GBM model. As time to delivery $T - t \to 0$, then all models give the same price since $F(T, T) = S(T)$.

We turn to the forward prices under stationary CAR models with three different stochastic processes driving the dynamics: Brownian motion, NIG and BNS stochastic volatility respectively formulated in Propositions 3.5–3.7. Under CAR model with Brownian motion increments, we may have fixed shape of forward by having summation of deterministic function scaled by some constant in the second term. However, the first term which includes $\mathbf{X}(t)$ is scaled by $\mathbf{e}_1' e^{A(T-t)}$, and gives rise to a stochastic path. This is similar to the forward price under two other CAR models. However, for forward under CAR with BNS stochastic volatility increments in Proposition 3.7, the

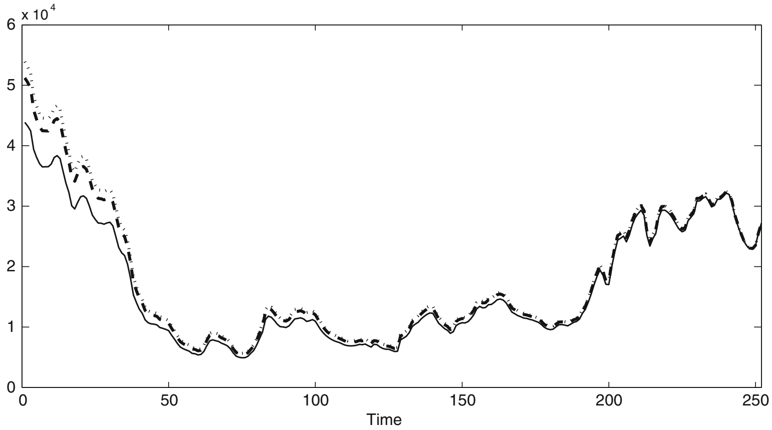

Fig. 2 Forward prices under GBM (*complete line*), NIG Lévy (*dashed line*) and BNS stochastic volatility (*dotted line*) spot models with $T = 252$ and $\theta = \theta_L = \theta_V = 0$

second term also contributes to the stochasticity with having time varying volatility, while other terms gives rise to a fixed shape.

To this end, let us investigate the following term

$$\mathbf{e}_1' e^{A(T-t)} \mathbf{X}(t), \tag{4.1}$$

which contributes to the stochastic variation of the forward in all CAR models. We may represent (4.1) as

$$\mathbf{e}_1' e^{A(T-t)} (e_1 \mathbf{x}_1(t) + e_2 \mathbf{x}_2(t) + \cdots + e_p \mathbf{x}_p(t)).$$

Each of these random factors is scaled by the function $m_i(x) = \mathbf{e}_1' e^{Ax} e_i$ for $i = 1, \ldots, p$ and $x = T - t$ or time to delivery. Note that when $x \to \infty$, the function $m_i(x)$ tends to zero since the real parts of the eigenvalues are all negative in stationarity. Furthermore, if $x = 0$ then $m_i(0) = 1$ for $i = 1$ and equal zero for $i \geq 2$. These can be observed in Fig. 3 using parameters for CAR(3) model in Benth et al. [8] where there are hump shapes contributed by the second and third factors. The level of the curve is depending on the different speed of mean reversions which are incorporated in the matrix A.

In Fig. 4, we show the plots of $m_i(x)$ for $i = 1, 2, 3$ and $T = 252$. Observe the nonnegativity of the function $m(x)$. The first curve which influences $\mathbf{x}_1(t)$ is downward sloping at very slow rate, while the second and third curves which affect $\mathbf{x}_2(t)$ and $\mathbf{x}_3(t)$ respectively show hump in the short end.

We come to the second term in Proposition 3.7 which also contributes to the stochastic path of the forward for CAR model with BNS stochastic volatility increments. There arise question on how weighting factor influences the stochastic volatility. Therefore, we investigate the shape of the term given by

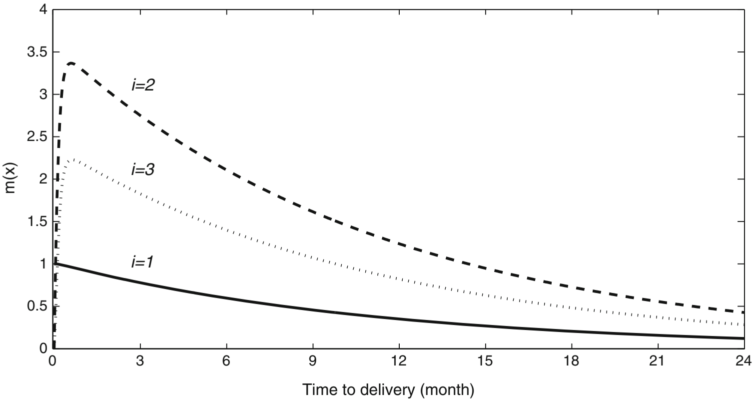

Fig. 3 The function $m(x)$ for CAR(3) model with $t = 0$

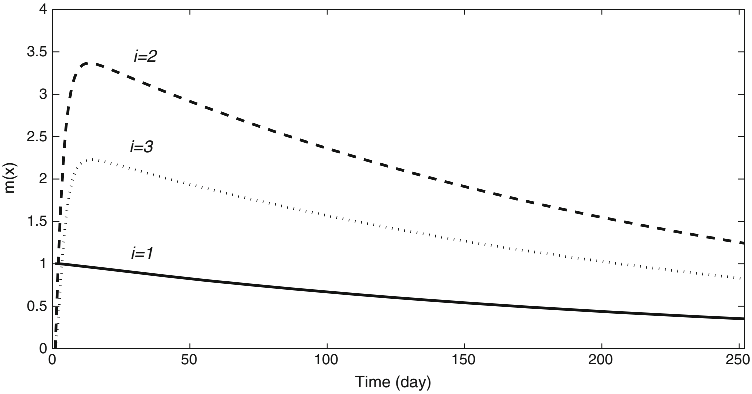

Fig. 4 The function $m(x)$ for CAR(3) model with $T = 252$

$$\int_t^T (\mathbf{e}_1' e^{A(T-u)} \mathbf{e}_p)^2 e^{-\lambda_j (u-t)}\, du, \tag{4.2}$$

which appears next to $V_j(t)$. We know that matrix A is diagonalizable since $A \in GL_p(\mathbb{R})$. Hence, we can represent $e^{A(T-u)} = \mathbf{v}e^{\mathbf{D}(T-u)}\mathbf{v}^{-1}$ where $\mathbf{v}$ is the matrix of eigenvectors and $\mathbf{D}$ is diagonal matrix with eigenvalues λ_k for $k = 1, \ldots, p$ as the diagonal entries. We can use the spectral representation to show that

$$\mathbf{e}_1' e^{A(T-u)} \mathbf{e}_p = \sum_{k=1}^{p} a_k e^{\lambda_k (T-u)} \mathbf{e}_1' \mathbf{v}_k, \tag{4.3}$$

for $a_k \in \mathbb{R}$. Note that the real part of eigenvalues are all negative from stationarity property of CAR process. Consider the special case of $p = 1$ where the matrix A is simply $-\alpha_1$. As $x = T - t \to \infty$ then the function (4.2) goes to zero for $\lambda_{j=1}$ is real. Obviously for $x = 0$, (4.2) becomes zero. This produces a hump shape in the short end

Fig. 5 The shape of (4.2) with $t = 0$, $\alpha_1 = 0.005$ and $\lambda_{j=1} = 0.5$

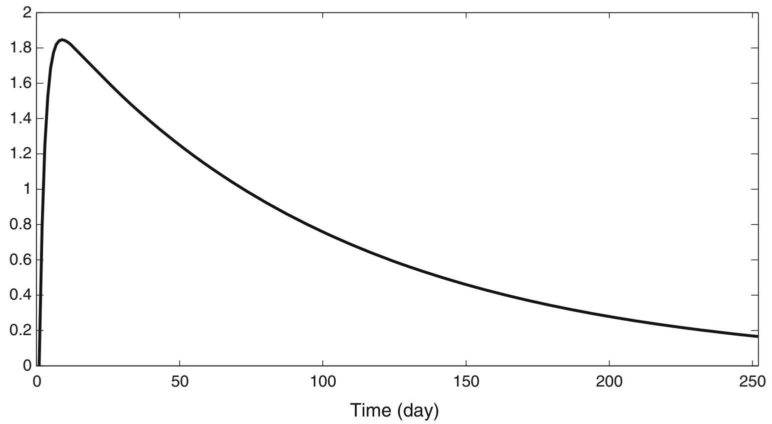

Fig. 6 The shape of (4.2) with $T = 252$, $\alpha_1 = 0.005$ and $\lambda_{j=1} = 0.5$

as illustrated in Fig. 5 where the shape is changing from contango to backwardation over time.

Now setting $T = 252$ and we observe the shape of (4.2) as time is moving forward until delivery. The shape is plotted in Fig. 6. We observe an upward trend in the short end of the curve and as time is increasing, the curve is decreasing towards some value.

As noted, if we let $x \to \infty$ then the curve will decrease towards zero. For the case of $p > 1$, the curve of (4.3) is in backwardation as long as eigenvalues have negative real parts and the constants a_k is not all with positive sign.

5 Conclusion

In this paper, we study the pricing of forward freight contracts under spot-forward relationship framework. We base our study on empirical investigation of spot freight rates in Benth et al. [8], where six different stochastic models were introduced to explain freight rates evolution in dry bulk market. Using no arbitrage pricing theory, we derive explicit forward price for all models discovered in the paper. In addition, we introduce a change of measure from physical probability to risk-neutral measure using Esscher transform. As a consequence, the market price of risk is incorporated into the forward formula and the market price of volatility risk is included in the model with stochastic volatility. For the case related to Brownian motion, the Esscher is nothing but a traditional Girsanov transform.

Further, we investigate the various shapes of forward curves based on our forward price formula. The curves for forward under geometric Brownian motion and exponential NIG Lévy spot models are fixed, in contrast with the curve under BNS stochastic volatility spot model because of presence of stochastic volatility factor. We also examine the shape of the forward curve under stationary CAR model with different Lévy increments. The impact of all factors goes through exponential function. While most of the terms contribute to a fixed shape of the forward, the state of CAR process and stochastic volatility provide fluctuations on the forward.

The findings in this article can be used for empirical study of the market forward price where possibly we can calculate the risk premium by minimizing the distance between theoretical and market forward price as suggested in Benth [6]. The risk premium calculation for weather market has been demonstrated by Hardle and Lopez-Cabrera [13]. Moreover, we can price the options based on explicit forward derived in this paper for future work.

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References

1. Barndorff-Nielsen, O.E.: Processes of normal inverse Gaussian type. Financ. Stoch. **2**(1), 41–68 (1998)
2. Barndorff-Nielsen, O.E., Shephard, N.: Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. J. R. Stat. Soc. B **63**(Part 2), 167–241 (2001)
3. Barndorff-Nielsen, O.E., Shiryaev, A.: Change of Time and Change of Measure. World Scientific, Singapore (2010)
4. Benth, F.E., Šaltytė, Benth J.: The normal inverse Gaussian distribution and spot price modelling in energy markets. Int. J. Theor. Appl. Financ. **7**(2), 177–192 (2004)
5. Benth, F.E., Šaltytė, Benth J., Koekebakker, S.: Stochastic Modelling of Electricity and Related Markets. World Scientific, Singapore (2008)
6. Benth, F.E.: The stochastic volatility model of Barndorff-Nielsen and Shephard in commodity markets. Math. Financ. **21**(4), 595–625 (2011)
7. Benth, F.E., Sgarra, C.: The risk premium and the Esscher transform in power markets. Stoch. Anal. Appl. **30**, 20–43 (2012)
8. Benth, F.E., Koekebakker, S., Taib, C.M.I.C.: Stochastic dynamical modelling of spot freight rates. IMA J. Manag. Math. (2014). doi:10.1093/imaman/dpu001
9. Borovkova, S., Geman, H.: Seasonal and stochastic effects in commodity forward curves. Rev. Deriv. Res. **9**, 167–186 (2006)
10. Brockwell, P.J.: Lévy-driven CARMA process. Ann. Inst. Stat. Math. **53**(1), 113–124 (2001)
11. Esscher, F.: On the probability function in the collective theory of risk. Skand. Aktuarietidskr. **15**, 175–195 (1932)

12. Gerber, H.U., Shiu, E.S.W.: Option pricing by Esscher transforms. Trans. Soc. Actuar. **46**, 99–191 (1994). (with discussion)
13. Hardle, W.K., Lopez Cabrera, B.: The implied market price of weather risk. Appl. Math. Financ. **19**(1), 59–95 (2009)
14. Kavussanov, M., Alizadeh, M.A.H.: Seasonality patterns in tanker spot freight rate markets. Econ. Model. **19**, 747–782 (2002)
15. Kavussanov, M., Nomikos, N.: The forward pricing function of the shipping freight futures market. J. Futur. Mark. **19**, 353–376 (1999)
16. Kavussanov, M., Nomikos, N.: Futures hedging when the structure of the underlying asset changes: the case of the BIFFEX contract. J. Futur. Mark. **20**, 776–801 (2000)
17. Koekebakker, S., Ådland, R.O.: Modelling forward freight rate dynamics-empirical evidence from time charter rates. Marit. Policy Manag. **31**(4), 319–335 (2004)
18. Prokopczuk, M.: Pricing and hedging in the freight futures market. J. Futur. Mark. **31**, 440–464 (2011)
19. Schwartz, E.S.: The stochastic behaviour of commodity prices: implications for valuation and hedging. J. Financ. **52**(3), 923–973 (1997)
20. Zhang, S., Zhang, X.: Exact simulation of IG-OU processes. Methodol. Comput. Appl. Probab. **10**, 337–355 (2008)